Period 4, Dec 12, 2024

Definitions of Trigonometric Functions of Any Angle

Let θ be any angle in standard position and let $P = (x, y)$ be a point on the terminal side of θ. If $r = \sqrt{x^2 + y^2}$ is the distance from $(0, 0)$ to (x, y), as shown in **Figure 4.43**, the **six trigonometric functions of θ** are defined by the following ratios:

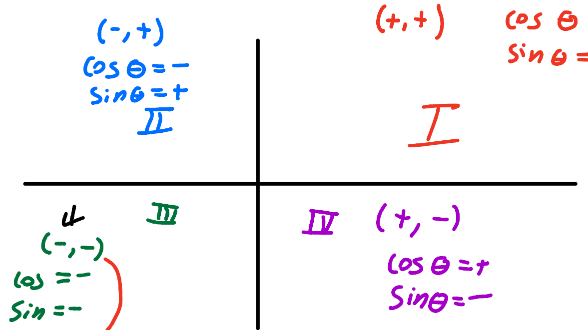
$$\sin\theta = \frac{y}{r} = \frac{\text{opp}}{\text{HYP}} \qquad \csc\theta = \frac{r}{y},\ y \neq 0 = \frac{\text{HYP}}{\text{opp}}$$

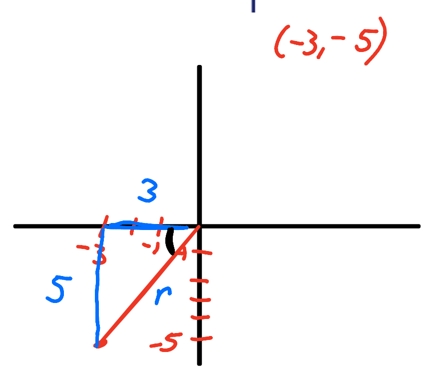
$$\cos\theta = \frac{x}{r} = \frac{\text{adj}}{\text{HYP}} \qquad \sec\theta = \frac{r}{x},\ x \neq 0 = \frac{\text{HYP}}{\text{adj}}$$

$$\tan\theta = \frac{y}{x},\ x \neq 0 = \frac{\text{opp}}{\text{adj}} \qquad \cot\theta = \frac{x}{y},\ y \neq 0. = \frac{\text{adj}}{\text{opp}}$$

The ratios in the second column are the reciprocals of the corresponding ratios in the first column.

Let $P = (-3, -5)$ be a point on the terminal side of θ. Find each of the six trigonometric functions of θ.





$(-3)^2 + (-5)^2 = 9 + 25 = 34 = r^2$

$\sqrt{34} = r$

$\sin\theta = -\frac{5}{\sqrt{34}} \cdot \frac{\sqrt{34}}{\sqrt{34}} = \frac{-5\sqrt{34}}{34}$

$\cos\theta = -\frac{3}{\sqrt{34}} \cdot \frac{\sqrt{34}}{\sqrt{34}} = \frac{-3\sqrt{34}}{34}$

$\csc\theta = \frac{1}{\sin\theta} = -\frac{\sqrt{34}}{5}$

$\sec\theta = \frac{1}{\cos\theta} = -\frac{\sqrt{34}}{3}$

$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{-5}{\sqrt{34}}}{\frac{-3}{\sqrt{34}}} = \frac{+5}{\cancel{\sqrt{34}}} \cdot \frac{\cancel{\sqrt{34}}}{\cancel{+}3} = \frac{5}{3}$

$\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{3}{5}$

Evaluate, if possible, the sine function and the tangent function at the following four quadrantal angles:

a. $\theta = 0° = 0$ **b.** $\theta = 90° = \frac{\pi}{2}$ **c.** $\theta = 180° = \pi$ **d.** $\theta = 270° = \frac{3\pi}{2}$.

Solution

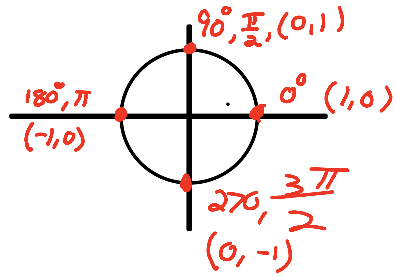
$\sin 0 = 0$

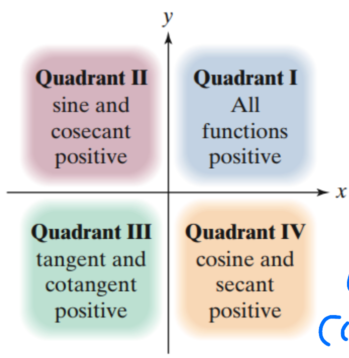
$\sin \frac{\pi}{2} = 1$

$\sin 180° = 0$

$\sin \pi = 0$

$\sin 270 = \sin \frac{3\pi}{2} = -1$





$(+, -)$

$(\cos\theta, \sin\theta)$

If $\tan\theta < 0$ and $\cos\theta > 0$, name the quadrant in which angle θ lies.

$\cot^2\theta + 1 = \csc^2\theta$

Given $\tan\theta = -\frac{2}{3}$ and $\cos\theta > 0$, find $\cos\theta$ and $\csc\theta$. B

Quad II, IV

Quad I, IV

θ is in IV Quad

$\cot^2\theta + 1 = \csc^2\theta$

$\left(-\frac{3}{2}\right)^2 + 1 = \csc^2\theta$

$\frac{9}{4} + \frac{4}{4} = \frac{13}{4} = \csc^2\theta$

$\sqrt{\csc^2\theta} = \sqrt{\frac{13}{4}}$

$\csc\theta = \pm\frac{\sqrt{13}}{2}$

$\csc\theta = -\frac{\sqrt{13}}{2}$

$\tan^2\theta + 1 = \sec^2\theta$

$\left(-\frac{2}{3}\right)^2 + 1 = \sec^2\theta$

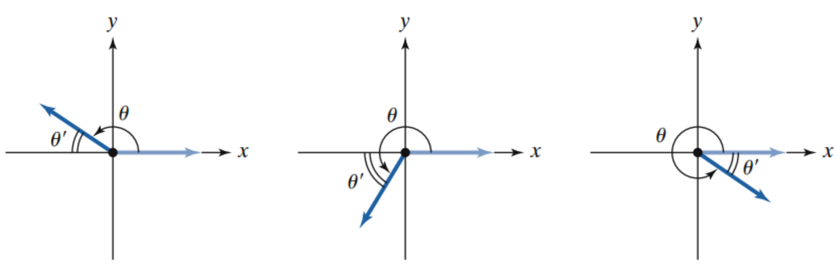
$\frac{4}{9} + \frac{9}{9} = \frac{13}{9} = \sec^2\theta = \frac{1}{\cos^2\theta}$

$\frac{13}{9} = \frac{1}{\cos^2\theta} \Rightarrow \sqrt{\cos^2\theta} = \sqrt{\frac{9}{13}} = \pm\frac{3}{\sqrt{13}} \cdot \frac{\sqrt{13}}{\sqrt{13}}$

$\cos\theta = \pm\frac{3\sqrt{13}}{13} = +\frac{3\sqrt{13}}{13}$

Definition of a Reference Angle

Let θ be a nonacute angle in standard position that lies in a quadrant. Its **reference angle** is the positive acute angle θ' formed by the terminal side of θ and the x-axis.



If $90° < \theta < 180°$, then $\theta' = 180° - \theta$.

If $180° < \theta < 270°$, then $\theta' = \theta - 180°$.

If $270° < \theta < 360°$, then $\theta' = 360° - \theta$.

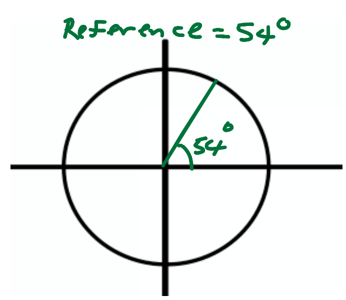
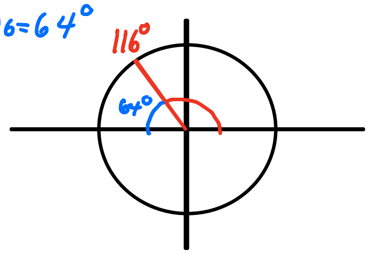


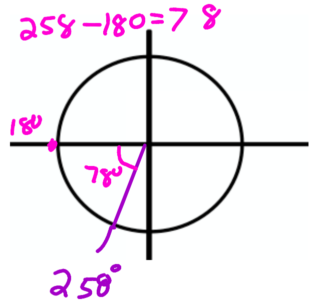
Figure 4.51 Reference angles, θ', for positive angles, θ, in quadrants II, III, and IV

Reference angle

180−116=64°

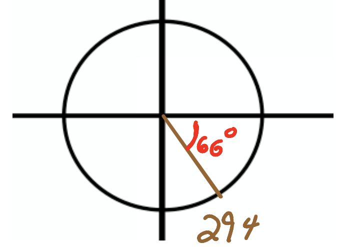


258−180=78



Reference angle

Refrence angle=66



360−294=66

Find the reference angle, θ', for each of the following angles:

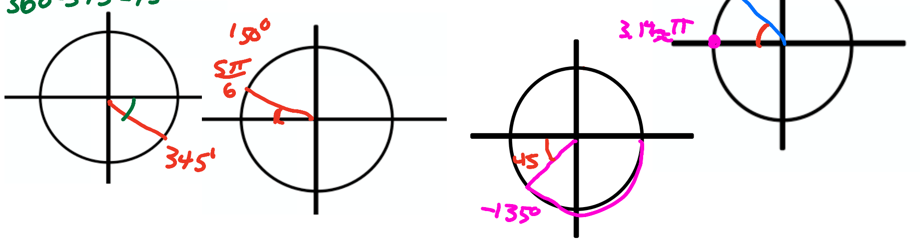
a. $\theta = 345°$ **b.** $\theta = \frac{5\pi}{6}$ **c.** $\theta = -135°$ **d.** $\theta = 2.5.$

$\theta' = 15°$ $\theta' = \frac{\pi}{6} = 30°$ $\theta = 45$ $\frac{\pi}{2} = 1.57$

$\theta' = \pi - 2.5$

$\theta' = .64159$

$360 - 345 = 15$



Finding Reference Angles for Angles Greater Than 360°(2π) or Less Than $-360°(-2\pi)$

1. Find a positive angle α less than 360° or 2π that is coterminal with the given angle.
2. Draw α in standard position.
3. Use the drawing to find the reference angle for the given angle. The positive acute angle formed by the terminal side of α and the x-axis is the reference angle.

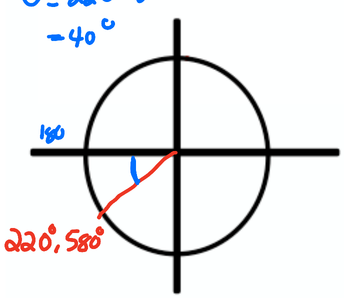
Find the reference angle for each of the following angles:

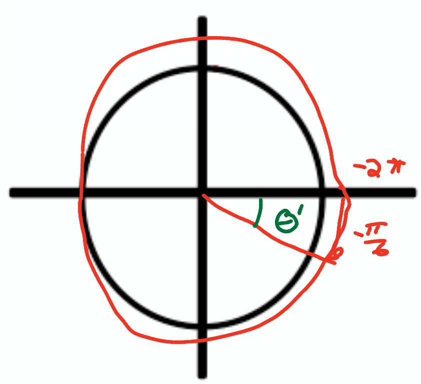
a. $\theta = 580°$ **b.** $\theta = \frac{8\pi}{3}$ **c.** $\theta = -\frac{13\pi}{6} = -2\frac{1}{6}\pi$

$\theta' = 30° = \frac{\pi}{6}$

$\theta' = 220 - 180$

$= 40°$

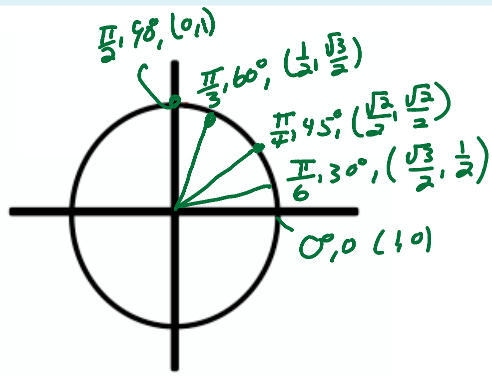




$580 - 360 = 220°$

Using Reference Angles to Evaluate Trigonometric Functions

The values of the trigonometric functions of a given angle, θ, are the same as the values of the trigonometric functions of the reference angle, θ', except possibly for the sign. A function value of the acute reference angle, θ', is always positive. However, the same function value for θ may be positive or negative.



Use reference angles to find the exact value of each of the following trigonometric functions:

a. $\sin 135°$ Quad II, sin +

$\theta' = 45$

$\sin 45 = \frac{\sqrt{2}}{2}$

$\sin 135° = +\frac{\sqrt{2}}{2}$

b. $\cos\frac{4\pi}{3}$ Quad III, cos −

$\cos\frac{\pi}{3} = \frac{1}{2}$

$\cos\frac{4\pi}{3} = -\frac{1}{2}$

c. $\cot\left(-\frac{\pi}{3}\right)$. Quad IV, cot −

$$\cot\frac{\pi}{3} = \frac{\cos\frac{\pi}{3}}{\sin\frac{\pi}{3}} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{\sqrt{3}} = \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$\cot\left(-\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{3}$$

CHECK POINT 7 Use reference angles to find the exact value of the following trigonometric functions:

a. $\sin 300°$ **b.** $\tan \frac{5\pi}{4}$ **c.** $\sec\left(-\frac{\pi}{6}\right)$.

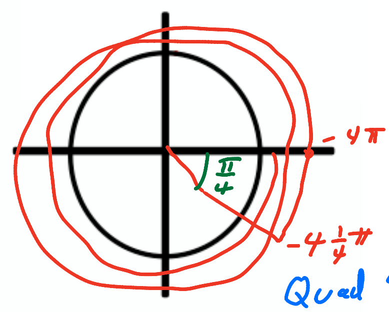
In our final example, we use positive coterminal angles less than 2π to find the reference angles.

EXAMPLE 8 **Using Reference Angles to Evaluate Trigonometric Functions**

Use reference angles to find the exact value of each of the following trigonometric functions:

a. $\tan \frac{14\pi}{3}$ **b.** $\sec\left(-\frac{17\pi}{4}\right)$.

Solution



$\frac{-17\pi}{4} = -4\frac{1}{4}\pi$

$\sec\frac{\pi}{4} = \frac{1}{\cos\frac{\pi}{4}} = \frac{1}{\frac{\sqrt{2}}{2}} = \frac{2}{\sqrt{2}}$

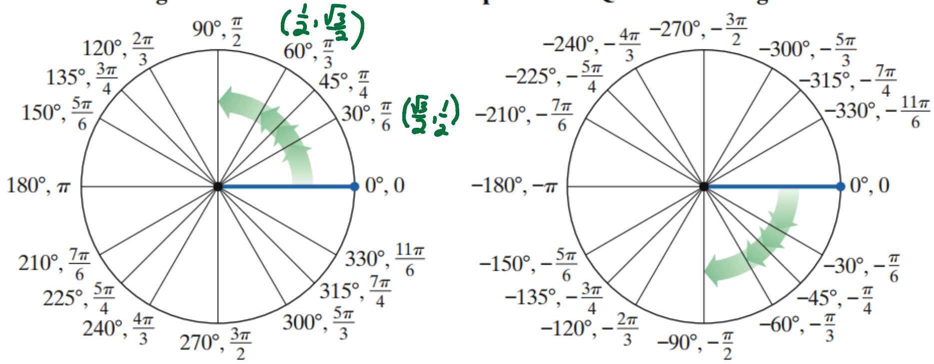
$\frac{2\sqrt{2}}{\sqrt{2}\sqrt{2}} = \frac{\cancel{2}\sqrt{2}}{\cancel{2}} = \sqrt{2}$

Quad 4

$\sec\theta = +$

$\sec\frac{-17\pi}{4} = \sec -\frac{\pi}{4} = \sqrt{2}$

Degree and Radian Measures of Special and Quadrantal Angles



Find the exact value of the expression. Do not use a calculator.

$\sin 87 = \cos(90-87) = \cos 3$

$\sin 3 = \cos(90-3) = \cos 87$

$\csc 3^\circ \sec 87^\circ - \tan 87^\circ \cot 3^\circ$

$$\frac{1}{\sin 3^\circ} \cdot \frac{1}{\cos 87} - \frac{\sin 87}{\cos 87} \cdot \frac{\cos 3}{\sin 3}$$

$\sin 87 = \cos 3$

$\sin 3 = \cos 87$

$$\frac{1}{\sin 3 \cos 87} - \frac{\sin 87 \cos 3}{\sin 3 \cos 87}$$

$\sin^2\theta + \cos^2\theta = 1$

$$\frac{1 - \sin 87 \cos 3}{\sin 3 \cos 87} = \frac{1 - \cos 3 \cdot \cos 3}{\sin 3 \cos 87} = \frac{1 - \cos^2 3}{\sin 3 \cos 87}$$

$$\frac{\sin^2 3 + \cancel{\cos^2 3} - \cancel{\cos^2 3}}{\sin 3 \cos 87}$$

$$\frac{\sin^{\cancel{2}} 3}{\cancel{\sin 3} \cos 87} = \frac{\sin 3}{\cos 87} = \frac{\cos 87}{\cos 87} = 1$$

$\sin 169 = 0.19080$

$\cos 169 = -0.9816$

$1 \approx 0.999 = (-0.9816)^2 + (0.19080)^2$

If $f(\theta) = 2 \sin\theta + \sin 2\theta$, find $f\left(\frac{\pi}{6}\right)$. Do not use a calculator and express each exact value as a single fraction.

$$F\left(\frac{\pi}{6}\right) = 2 \cdot \sin\frac{\pi}{6} + \sin 2 \cdot \frac{\pi}{6} = 2\sin\frac{\pi}{6} + \sin\frac{\pi}{3}$$

$$2 \cdot \frac{1}{2} + \frac{\sqrt{3}}{2} = \frac{2}{2} + \frac{\sqrt{3}}{2} = \frac{2+\sqrt{3}}{2} = 1 + \frac{\sqrt{3}}{2}$$

Use a calculator to find the value of the acute angle θ in radians.

$30^\circ = \frac{\pi}{6}$

$\tan\theta = 0.1233$

$\tan^{-1} 0.1233 = 0.12268$

If θ is an acute angle and **cos** $\theta = \frac{1}{6}$, find **csc** $\left(\frac{\pi}{2} - \theta\right)$.

$$\csc\left(\frac{\pi}{2} - \theta\right) = \frac{1}{\sin\left(\frac{\pi}{2} - \theta\right)} = \frac{1}{\cos\theta} = \frac{\frac{1}{1}}{\frac{1}{6}} = \frac{1}{1} \cdot \frac{6}{1} = 6$$

$\sin(90^\circ - \theta) = \cos\theta$

$\cos(90^\circ - \theta) = \sin\theta$

$\sin(\frac{\pi}{2} - \theta) = \cos\theta$

$\cos(\frac{\pi}{2} - \theta) = \sin\theta$

Find the exact value of the expression. Do not use a calculator.

$$\frac{\tan\frac{\pi}{4}}{2} + \frac{1}{\sec\frac{\pi}{3}}$$

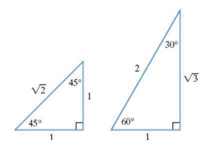
$$\text{Tan}\,\frac{\pi}{4} = \frac{\sin\frac{\pi}{4}}{\cos\frac{\pi}{4}} = \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = 1$$

$$\frac{1}{2} + \frac{1}{2} = \frac{2}{2} = 1$$

$$\text{Sec}\,\frac{\pi}{3} = \frac{1}{\cos\frac{\pi}{3}} = \frac{1}{\frac{1}{2}} = 2$$

Use the triangles given on the right to evaluate the expression given below. If necessary, express the value without a square root in the denominator by rationalizing the denominator.

$$\sin\frac{\pi}{6}\cos\frac{\pi}{4} - \tan\frac{\pi}{3}$$



$$\text{Tan}\,\frac{\pi}{3} = \frac{\sin\frac{\pi}{3}}{\cos\frac{\pi}{3}} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \frac{\sqrt{3}}{2}\cdot\frac{2}{1} = \sqrt{3}$$

$$\text{Sin}\,\frac{\pi}{6}\cdot\text{Cos}\,\frac{\pi}{4} - \text{Tan}\,\frac{\pi}{3}$$

$$\frac{1}{2}\cdot\frac{\sqrt{2}}{2} - \sqrt{3}$$

$$\frac{\sqrt{2}}{4} - \sqrt{3}$$

$$\frac{3\cdot\pi}{3\cdot 2} - \frac{\pi}{6} = \frac{3\pi - \pi}{6} = \frac{2\pi}{6} = \frac{\pi}{3}$$

Find a cofunction with the same value as the given expression.

$\tan\frac{\pi}{6} = \cot\left(\frac{\pi}{2}-\frac{\pi}{6}\right) = \cot\frac{\pi}{3}$

The answer is $\square$.

(Simplify your answer. Type an exact answer in terms of π. Use integers or fractions for any numbers in the expression.)

Use the given function to complete parts (a) through (e) below.

$f(x) = x^4 - 16x^2$

$F(x) = x^2(x^2-16) = x^2(x^2-4^2) = x^2(x-4)(x+4)$

Zero's $\Rightarrow$ $x^2=0$ or $x-4=0$ or $x+4=0$

$x=0$ $x=0$ $x=4$ $x=-4$

multiply 2 Even Bounces

multiply 1 odd Crosses

At which zeros does the graph of the function cross the x-axis? Select the correct choice below and, if necessary, fill in the answer box to complete your choice.

$x=4$ or $x=-4$

At which zeros does the graph of the function touch the x-axis and turn around? Select the correct choice below and, if necessary, fill in the answer box to complete your choice.

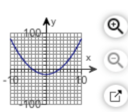
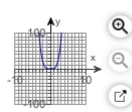
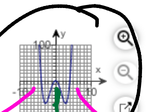
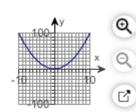
bounce $x=0$

d) Determine the symmetry of the graph.

$F(x) = x^4 - 16x^2$ $F(-x) = (-x)^4 - 16(-x)^2 = x^4 - 16x^2$

$F(x) = F(-x)$ Even Sym over y axis

e) Determine the graph of the function.

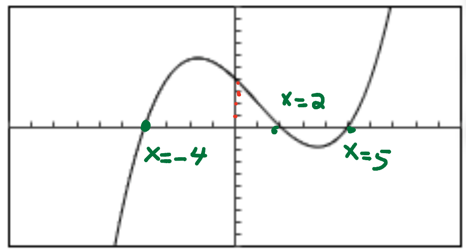
Cross $x=-4$

Bounce $x=0$

Cross $x=4$

The graph to the right is a complete graph, that is, it is continuous and displays the function's end behavior. All zeros are integers. Answer the following questions.

X gets Big
Y gets Big



[− 10,10,1] by [− 100,100,10]

Zeros are
$\underbrace{x=-4,2,5}_{\text{once}}$

(a) List the zeros whose multiplicity is even. Select the correct choice below and fill in any answer boxes within your choice.

None

List the zeros whose multiplicity is odd. Select the correct choice below and fill in any answer boxes within your choice.

$-4, 2, 5$

(b) Write an equation, expressed as the product of factors, of a polynomial that the graph might represent. Use a leading coefficient of 1 or − 1, and make the degree of f as small as possible.

$F(x) = +(x+4)(x-2)(x-5)$

What is the y-intercept of the graph?

$F(0) = (0+4)(0-2)(0-5)$
$4 \cdot 2 \cdot 5 = 40$

Divide using synthetic division.

$\left(2x^2 - 14x - 7x^3 + x^4\right) \div (7 + x)$

$x^4 - 7x^3 + 2x^2 - 14x$

-7	1	-7	2	-14	0
		-7	98	-700	4998
	1	-14	100	-714	$\boxed{4998}$

$1x^3 - 14x^2 + 100x - 714 + \frac{4998}{7+x}$

In the following problem, divide using long division. State the quotient, q(x), and the remainder, r(x).

$(x^2 + 8x + 12) \div (x + 2)$

$$
\begin{array}{r}
x+6 \\
x+2 \,\overline{)\, x^2+8x+12} \\
-(x^2+2x) \downarrow \\
\hline
0 \quad 6x+12 \\
-(6x+12) \\
\hline
0+0
\end{array}
$$

$x+6+\frac{0}{x+2}$